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Casimir operators of the exceptional group G_2

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Abstract

We calculate the degree 2 and 6 Casimir operators of G_2 in explicit form, with the generators of G_2 written in terms of the subalgebra A_2.

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I. INTRODUCTION

The general algorithm for obtaining invariant quantities in semi-simple groups goes back to Killing [1], Cartan [2], and Racah [3]. Killing's procedure is based on the characteristic equation defined by the adjoint representation. Racah extended this idea using any representation, and subsequent work showed that a complete set of invariants can be obtained in this way [4,5].

Our motivation for the work presented here are the invariants of the exceptional group E_6 which are of particular interest to particle physics. To set up the method, it is illustrative to look at the Casimir operators of G_2 using generators in terms of the subalgebra A_2. (The group E_6 has the subalgebra $A_2 \oplus A_2 \oplus A_2$.)

II. GENERATORS OF G_2 IN A_2 BASIS

We identify the 14 generators of G_2 in terms of the A_2 subalgebra. Corresponding to the reduction

$$\mathbf{14} = \mathbf{8} + \mathbf{3} + \bar{\mathbf{3}} \tag{2.1}$$

under the restriction of G_2 to A_2 we label the generators as ${g_k}^\ell$, a_k, b^ℓ with all indices ranging from 1 to 3. We have the commutation relations [6,7]

$$\left[{g_k}^\ell,\ {g_m}^n\right] = {\delta_m}^\ell {g_k}^n - {\delta_k}^n {g_m}^\ell \tag{2.2}$$

$$\left[{g_k}^\ell,\ a_m\right] = {\delta_m}^\ell a_k - \frac{1}{3}{\delta_k}^\ell a_m \tag{2.3}$$

$$\left[{g_k}^\ell,\ b^n\right] = -{\delta_k}^n b^\ell + \frac{1}{3}{\delta_k}^\ell b^n \tag{2.4}$$

$$\left[a_m,\ b^n\right] = {g_m}^n \tag{2.5}$$

$$\left[a_m,\ a_n\right] = -\frac{2}{\sqrt{3}}\epsilon_{mn\ell} b^\ell \tag{2.6}$$

$$\left[b^m,\ b^n\right] = \frac{2}{\sqrt{3}}\epsilon^{mn\ell} a_\ell \tag{2.7}$$

Equation (2.2) together with the constraint

$$\sum_k g_k{}^k = 0 \tag{2.8}$$

identifies $g_k{}^\ell$ as the octet of generators generating A_2. Equation (2.3) states that the three generators a_m transform as a triplet, and Eq. (2.4) states that the b^n transform as an antitriplet under the A_2 generated by the $g_k{}^\ell$. These properties can also be read off the root diagram shown in Fig. 1. Lastly we note that in a unitary representation the hermiticity properties are given by

$$g_k{}^{\ell\dagger} = g_\ell{}^k, \tag{2.9}$$

$$a_m{}^\dagger = b^m. \tag{2.10}$$

III. THE MATRIX Q

Consider an arbitrary finite-dimensional semisimple Lie algebra L and denote its generators by $X_\mu,\ \mu = 1, 2, \ldots, d$, where $d = \dim L$. Consider next some nontrivial representation and denote the generators in that represention by x_μ. Because L is semisimple it follows that the $d \times d$ symmetric matrix $g_{\mu\nu}$ defined by

$$g_{\mu\nu} = tr(x_\mu x_\nu) \tag{3.1}$$

is nonsingular. Hence we can define

$$g^{\mu\nu} = \left(g^{-1}\right)_{\mu\nu} \tag{3.2}$$

and introduce the matrix Q by

$$Q = X_\mu \otimes x^\mu = g^{\mu\nu} X_\mu \otimes x_\nu. \tag{3.3}$$

We also can define the $(p+1)$-th power of Q $(p \geq 0)$ by

$$Q^{p+1} = QQ^p \tag{3.4}$$

with the property

$$[X_\mu,\ tr\, Q^p] = 0 \tag{3.5}$$

so that degree p Casimir operators can be taken equal to $tr\, Q^p$. The procedure just described is essentially that of Gruber and O'Raifeartaigh [4] where the proof of Eq. (3.5) can be found.

We now construct Q in the case of $L = G_2$. The nontrivial representation is chosen as the 7-dimensional irreducible representation of G_2 (the smallest irreducible representation of G_2). Thus Q is a 7×7 matrix whose matrix elements are proportional to the 14 generators of G_2. Since the $\mathbf{7}$ of G_2 decomposes under A_2 as

$$\mathbf{7} = \mathbf{3} \oplus \mathbf{1} \oplus \bar{\mathbf{3}} \tag{3.6}$$

the 7×7 matrix Q naturally breaks up into a 3×3 matrix given as

$$Q = \begin{pmatrix} G & \alpha & B \\ \beta^T & 0 & -\alpha^T \\ A & -\beta & -G^T \end{pmatrix}. \tag{3.7}$$

Here $A,\ B,\ G,\ G^T$ are 3×3 matrices, $\alpha,\ \beta$ are 3×1 (column) matrices, and $\alpha^T,\ \beta^T$ are 1×3 (row) matrices. Explicitly, we have

$$\left.\begin{array}{r} G_{k\ell} = {g_k}^\ell, \quad (G^T)_{k\ell} = {g_\ell}^k \\ A_{k\ell} = -\frac{1}{\sqrt{3}}\epsilon^{k\ell r}a_r,\ B_{k\ell} = \frac{1}{\sqrt{3}}\epsilon_{k\ell r}b^r \\ \alpha_k = \sqrt{\frac{2}{3}}a_k, \quad \beta_k = \sqrt{\frac{2}{3}}b^k \end{array}\right\} \tag{3.8}$$

Except for some renumbering of rows and columns the representation of the generators of G_2 by 7×7 matrices used in Eq. (3.7) is precisely the same as given by Patera [8] and it agrees with Berdjis [5] and Ekins and Cornwell [9] but for different normalization of the generators.

We write the 7×7 matrix Q^p as

$$Q^p = \begin{pmatrix} G_p & \alpha_p & B_p \\ \beta_p^T & c_p & \gamma_p^T \\ A_p & \rho_p & H_p \end{pmatrix} \tag{3.9}$$

where $A_p,\ B_p,\ G_p\ H_p$ are 3×3 matrices, $\alpha_p,\ \beta_p,\ \gamma_p,\ \rho_p$ are 3×1 matrices, and c_p is a 1×1 matrix. Eq. (3.4) implies

$$\begin{aligned}
G_{p+1} &= GG_p + \alpha\beta_p^T + BA_p, \\
\alpha_{p+1} &= G\alpha_p + \alpha c_p + B\rho_p, \\
B_{p+1} &= GB_p + \alpha\gamma_p^T + BH_p, \\
\beta_{p+1}^T &= \beta^T G_p - \alpha^T A_p, \\
c_{p+1} &= \beta^T \alpha_p - \alpha^T \rho_p, \\
\gamma_{p+1}^T &= \beta^T B_p - \alpha^T H_p, \\
A_{p+1} &= AG_p - \beta\beta_p^T - G^T A_p, \\
\rho_{p+1} &= A\alpha_p - \beta c_p - G^T \rho_p, \\
H_{p+1} &= AB_p - \beta\gamma_p^T - G^T H_p.
\end{aligned} \tag{3.10}$$

These recursive definitions enable us to investigate the explicit form of the Casimir operators of G_2 in the next section.

IV. THE CASIMIR OPERATORS $\mathcal{C}_p$

It follows from the preceding that the degree p Casimir operator $\mathcal{C}_p$ is given by

$$\mathcal{C}_p \equiv trQ^p = c_p + tr(G_p + H_p) \tag{4.1}$$

which by repeated use of Eq. (3.10) can be explicitly exhibited as a homogeneous polynomial of degree p in the matrix elements of Q, i.e., the generators of G_2.

Functionally independent Casimir operators are in fact only two: $\mathcal{C}_2$ and $\mathcal{C}_6$. This is because the number of independent Casimir operators equals the rank of the group and their degrees obey the relation $p = m+1$, where m are the so-called exponents of the group. For G_2 we have that rank $= 2$ and $m = 1$ and 5.

In our formulation we find the functionally independent Casimir operators as follows: Since Q is a 7×7 matrix, all Casimirs of degree $p > 7$ are, by the Cayley-Hamilton theorem, not independent of Casimirs of lower degree. Further, Q carries an antisymmetry property corresponding to the fact that all irreducible representations of G_2 are orthogonal—as a consequence all odd degree Casimirs are expressible in terms of lower degree (and ultimately even degree) Casimirs. Thus we need only evaluate $\mathcal{C}_2$, $\mathcal{C}_4$ and $\mathcal{C}_6$, and we will find that $\mathcal{C}_4$ is expressible in terms of $\mathcal{C}_2$.

For $p = 2$ Eq. (4.1) becomes

$$\begin{aligned}\mathcal{C}_2 &= c_2 + tr(G_2 + H_2)\\ &= \beta^T\alpha + \alpha^T\beta + tr(GG + \alpha\beta^T + BA + AB + \beta\alpha^T + G^TG^T)\\ &= 2\left\{\langle g^2\rangle + a\cdot b + b\cdot a\right\}\\ &= 2\left\{\langle g^2\rangle + 2b\cdot a\right\}\end{aligned}\tag{4.2}$$

where we used the following identities and definitions:

$$\begin{aligned}\alpha^T\beta = \beta^T\alpha &= \frac{2}{3}b^k a_k \equiv \frac{2}{3}b\cdot a,\\ trG^TG^T = trGG &= g_k{}^\ell g_\ell{}^k \equiv \langle g^2\rangle,\\ tr\beta\alpha^T = tr\alpha\beta^T &= \frac{2}{3}a_k b^k \equiv \frac{2}{3}a\cdot b\\ trAB = trBA = B_{k\ell}A_{\ell k} &= \frac{1}{3}\epsilon_{k\ell r}b^r\epsilon^{k\ell s}a_s = \frac{2}{3}b\cdot a.\end{aligned}\tag{4.3}$$

We remark that $\langle g^2\rangle$ and $a\cdot b = b\cdot a$ are the *only* quadratic structures that can be formed out of the octet $g_k{}^\ell$, the triplet a_k and the antitriplet b^k that are SU(3) invariants—hence the quadratic Casimir of G_2 must be some linear combination of $\langle g^2\rangle$ and $b\cdot a$. Our second remark is that, since $a\cdot b = b\cdot a$, the quadratic Casimir is a *symmetric* polynomial in the generators, homogeneous of degree 2.

For $p = 4$ we have from Eq. (4.1)

$$\begin{aligned}\mathcal{C}_4 &= c_4 + tr(G_4 + H_4)\\ &= tr(G^4 + \text{transpose})\\ &\quad + tr\left(\frac{1}{2}ABAB + \alpha\beta^T\alpha\beta^T + \text{transpose}\right)_{\text{cycle 2}}\end{aligned}$$

$$+\, tr\left(G^2\alpha\beta^T + G^2BA - \frac{1}{2}GBG^TA + GB^T\beta\beta^T \right.$$
$$\left. + A^TG\alpha\alpha^T + BA\alpha\beta^T + \frac{1}{2}\alpha\alpha^T\beta\beta^T + \text{transpose}\right)_{\text{cycle 4}} \tag{4.4}$$

where the "transpose" instruction means: add the transpose of all the preceding terms in the bracket. The "cycle" instruction means: add all cyclic permutations. (Factors $\frac{1}{2}$ were inserted to take care of those cyclic terms that are equal to their own transpose.) Thus e.g.

$$G^4 + \text{transpose} \equiv G^4 + G^{T4}, \tag{4.5}$$

$$\begin{aligned} tr(\alpha\beta^T\alpha\beta^T)_{\text{cycle 2}} &\equiv (\alpha_a\beta_b\alpha_b\beta_a)_{\text{cycle 2}} \\ &\equiv \alpha_a\beta_b\alpha_b\beta_a + \beta_b\alpha_b\beta_a\alpha_a, \end{aligned} \tag{4.6}$$

$$\begin{aligned} tr(BA\alpha\beta^T)_{\text{cycle 4}} &\equiv (B_{ab}A_{bc}\alpha_c\beta_a)_{\text{cycle 4}} \\ &\equiv B_{ab}A_{bc}\alpha_c\beta_a + A_{bc}\alpha_c\beta_aB_{ab} + \alpha_c\beta_aB_{ab}A_{bc} + \beta_aB_{ab}A_{bc}\alpha_c. \end{aligned} \tag{4.7}$$

Eq. (4.4) exhibits $\mathcal{C}_4$ as a homogeneous polynomial of degree 4 in the generators of G_2. Alternatively, using commutation relations we can combine terms containing the same generators in different orders, and we arrive at an expression for $\mathcal{C}_4$ that shows that it is a function of $\mathcal{C}_2$:

$$\mathcal{C}_4 = \mathcal{C}_2\left[\frac{1}{4}\mathcal{C}_2 + \frac{14}{3}\right]. \tag{4.8}$$

This result includes the fact that

$$G^3 = 3G^2 + \left(\frac{1}{3}trG^3 - trG^2\right)\mathbf{1} + \left(\frac{1}{2}trG^2 - 2\right)G, \tag{4.9}$$

which follows from the generalized Cayley-Hamilton theorem.

We finally consider $p = 6$. This Casimir operator will be given in two different forms. In the first form we exhibit it explicitly as a homogeneous polynomial of degree 6 in the generators possessing cyclic symmetry, by which we mean that if a product of 6 generators appears in a certain order then so do all the cyclic permutations of that order. We exploit this cyclic property in the notation (as was already seen for $p = 4$) to write out fairly economically the 416 terms that arise.

In the second form we use commutation relations to relate terms that only differ in the order in which the generators appear. Here we end up with fewer terms but at the cost of having a polynomial of degree six which is no longer homogeneous.

To keep track of the 416 terms in the first form we group them by the degree in G and G^T. Thus, $\mathcal{C}_6$ is a sum of:

the 2 terms of degree 6 in G and G^T:

$$tr(G^6 + \text{transpose}) \tag{4.10}$$

plus the $7 \cdot 6$ terms of degree 4 in G and G^T:

$$tr\left(G^4\alpha\beta^T + G^4BA + G^3BG^TA^T + \frac{1}{2}G^2BG^{T2}A + \text{transpose}\right)_{\text{cycle 6}} \tag{4.11}$$

plus the $8 \cdot 6$ terms of degree 3 in G and G^T:

$$tr(G^3\alpha\alpha^TA^T + G^2\alpha\alpha^TG^TA + G^3B^T\beta\beta^T + G^2BG^T\beta\beta^T + \text{transpose})_{\text{cycle 6}} \tag{4.12}$$

plus the $21 \cdot 6 + 4 \cdot 3 = 138$ terms of degree 2 in G and G^T:

$$\begin{aligned}
&tr(G^2\alpha\beta^T\alpha\beta^T + G^2BA\alpha\beta^T + G^2\alpha\beta^TBA + G^2BABA \\
&\quad +G^2\alpha\alpha^T\beta\beta^T + G^2B\beta\alpha^TA + GBAG\alpha\beta^T + GBG^TA^T\alpha\beta^T \\
&\quad +G\alpha\beta^TBG^TA^T + \frac{1}{2}GBA^TBG^TA + \frac{1}{2}GBG^TABA^T \\
&\quad +\frac{1}{2}G\alpha\alpha^TG^T\beta\beta^T + \text{transpose})_{\text{cycle 6}} \\
&+tr(G\alpha\beta^TG\alpha\beta^T + GBAGBA + \text{transpose})_{\text{cycle 3}}
\end{aligned} \tag{4.13}$$

plus the $20 \cdot 6$ terms of degree 1 in G and G^T:

$$\begin{aligned}
&tr(G\alpha\beta^T\alpha\alpha^TA^T + G\alpha\alpha^T\beta\alpha^TA^T + G\alpha\alpha^TA^T\alpha\beta^T \\
&\quad +GB^T\beta\beta^T\alpha\beta^T + GB^T\beta\alpha^T\beta\beta^T + G\alpha\beta^TB^T\beta\beta^T \\
&\quad +GB^TA\alpha\alpha^TA + G\alpha\alpha^TAB^TA + GB\beta\beta^TBA^T \\
&\quad +GBA^TB\beta\beta^T + \text{transpose})_{\text{cycle 6}}
\end{aligned} \tag{4.14}$$

plus lastly the $3 \cdot 2 + 2 \cdot 3 + 9 \cdot 6 = 66$ terms of degree 0 in G and G^T:

$$
\begin{aligned}
&tr(\frac{1}{2}ABABAB + \alpha\beta^T\alpha\beta^T\alpha\beta^T + \text{transpose})_{\text{cycle 2}} \\
&+tr(\beta\beta^T B\beta\beta^T B + \alpha\alpha^T A\alpha\alpha^T A)_{\text{cycle 3}} \\
&+tr(\alpha\beta^T\alpha\beta^T BA + \alpha\alpha^T\beta\beta^T BA + \alpha\beta^T BABA \\
&\quad +\alpha\alpha^T\beta\alpha^T\beta\beta^T + \frac{1}{2}\alpha\beta^T B\beta\alpha^T A + \text{transpose})_{\text{cycle 6}}
\end{aligned} \tag{4.15}
$$

On the other hand, using the definitions given in Eq. (4.3) and applying commutation relations, we can rewrite $\mathcal{C}_6$ as follows [10]:

$$
\begin{aligned}
\mathcal{C}_6 = &\frac{44}{9}a{\cdot}b\,a{\cdot}b\,a{\cdot}b + \frac{10}{3}a{\cdot}b\,a{\cdot}b\langle g^2\rangle - 2b{\cdot}g{\cdot}a\,b{\cdot}g{\cdot}a \\
&+8b{\cdot}g{\cdot}g{\cdot}a\,a{\cdot}b + 5a{\cdot}b\langle g^2\rangle\langle g^2\rangle - 4b{\cdot}g{\cdot}g{\cdot}a\langle g^2\rangle \\
&+4b{\cdot}g{\cdot}a\langle g^3\rangle + \frac{2}{3}\langle g^3\rangle\langle g^3\rangle + \frac{1}{2}\langle g^2\rangle\langle g^2\rangle\langle g^2\rangle \\
&-\frac{8}{\sqrt{3}}\epsilon^{ijk}a_i(g{\cdot}a)_j(g{\cdot}g{\cdot}a)_k - \frac{8}{\sqrt{3}}\epsilon_{ijk}b^i(b{\cdot}g)^j(b{\cdot}g{\cdot}g)^k \\
&-20a{\cdot}b\,b{\cdot}g{\cdot}a - \frac{20}{3}\langle g^3\rangle a{\cdot}b + \frac{22}{\sqrt{3}}\langle g^2\rangle b{\cdot}g{\cdot}a - 2\langle g^2\rangle\langle g^3\rangle \\
&-\frac{112}{9}b{\cdot}g{\cdot}g{\cdot}a + 88a{\cdot}b\langle g^2\rangle + \frac{836}{9}a{\cdot}b\,a{\cdot}b + 21\langle g^2\rangle\langle g^2\rangle \\
&+\frac{5208}{81}b{\cdot}g{\cdot}a - \frac{40}{3}\langle g^3\rangle \\
&+\frac{584}{81}a{\cdot}b + \frac{3188}{27}\langle g^2\rangle.
\end{aligned} \tag{4.16}
$$

Depending on the application one might prefer the version given by Eqs. (4.10-4.15); although there are 416 terms they are all of the same degree and exhibit cyclic symmetry. The version given by Eq. (4.16) on the other hand has only 23 terms but is no longer homogeneous of degree 6 in the generators and no longer displays its explicit symmetry in the generators.

Previously, the only other fully explicit expression for $\mathcal{C}_6$ in the literature has been given by Hughes and Van der Jeught [11]. Their generators of G_2 are given in the $A_1 \oplus A_1$ basis (in contrast to our use of the A_2 basis) and they obtain for $\mathcal{C}_6$ an expression involving 29 terms. The $A_1 \oplus A_1$ basis, however, has the disadvantage of not displaying as much symmetry in the generators. Our motivation is to find explicit expressions for the Casimirs of E_6, and we expect the subalgebra $A_2 \oplus A_2 \oplus A_2$ to give the simplest results. The studies of G_2 in terms of A_2 seem to foster these hopes.

Lastly we should like to mention the work of Englefield and King [12], in which explicit expressions are given for the *eigenvalues* of all the Casimir operators of G_2.

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REFERENCES

[1] V. Killing, Math. Ann. **31**, 252 (1988).

[2] E. Cartan, Thesis p. 154

[3] G. Racah, Rend. Lincei **8**, 108 (1950).

[4] B. Gruber, and L. O'Raifeartaigh, J. Math. Phys. **5**, 1796 (1964).

[5] F. Berdjis, J. Math. Phys. **22**, 1851 (1981).

[6] M. Günaydin, and F. Gürsey, J. Math. Phys. **14**, 1651 (1973).

[7] R. LeBlanc, and D.J. Rowe, J. Math. Phys. **29**, 767 (1988).

[8] J. Patera, J. Math. Phys. **11**, 3027 (1970).

[9] J.M. Ekins, and J.F. Cornwell, Rep. Math. Phys. **7**, 167 (1973).

[10] To carry out this step of our calculation we used the algebraic programming system REDUCE.

[11] J.W.B. Hughes, and J. Van der Jeugt, J. Math. Phys. **26**, 894 (1985).

[12] M.J. Englefield and R.C. King, J. Phys. A **13**, 2297 (1980).

FIGURES

FIG. 1. The root diagram of G_2 in A_2 basis.